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Polynomial Solutions for Coupled $U(1)$-Gauge Einstein Equations

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We point out numerical solutions in polynomial form to the field equations derived by Garfinkle and Laguna for $U(1)$-gauge cosmic strings. With these solutions we evaluate the second-order term for the deficit angle produced by the string.

1. INTRODUCTION

A cosmic string is a one-dimensional energy distribution defined by an extreme length as well as a great linear density. Models to describe elementary particle physics by means of gauge theories with spontaneous symmetry breaking have given rise to an intensive study of this kind of string.

An infinite-length cosmic string is a static, cylindrically symmetric configuration of a self-interacting scalar field which is minimally coupled to a gauge field. Since this string has stress energy, it couples to the gravitational field and its gravitational effects are calculable by Einstein's equations (see, for instance, Zel'dovich, 1980; Vilenkin, 1981*a*; Linde, 1979, and references therein).

This subject has been considered by many authors. Vilenkin (1981*b*) studied the gravitational properties of vacuum domain walls and strings in the linear approximation of general relativity. Gott (1985) obtained the exact interior and exterior solutions to Einstein's equations, for vacuum strings. Garfinkle (1985) investigated the properties of infinite-length cosmic strings by considering the full coupled equations for the metric, the scalar, and $U(1)$ gauge fields. Laguna-Castilho and Matzner (1987*a*) proposed an approach for an infinite-length $U(1)$ cosmic string as a cylindrical and singular shell enclosing a region of false vacuum. They pointed out the consistency of this

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model with the full coupled equations for the metric and the scalar and gauge fields in curved space-time. Laguna-Castilho and Matzner (1987*b*) obtained numerical solutions for this approach. Garfinkle and Laguna (1989) considered the contribution of the gravitational self-interaction to the mass linear density μ and to the deficit angle $\Delta\phi$ of a long, static, straight cosmic string.

In this paper we consider Einstein-scalar-gauge equations derived by Garfinkle and Laguna (1989) and point out their solutions by numerical techniques. In a different way, we expand the corrections to $\Delta\phi$ in polynomial forms of the fields to find its second-order terms. In Section 2 we review briefly the formalism for self-gravitating $U(1)$-gauge cosmic strings and in Section 3 we present numerical results and conclusions.

2. STANDARD COSMIC STRING LAGRANGIAN AND FIELD EQUATIONS

We consider cosmic strings that consist of a $U(1)$-gauge vector field A_a and a complex scalar field $\Phi = R(\rho)\, e^{i\psi}$, with a Lagrangian ($G=c=\hbar=1$)

$$\mathscr{L} = -\tfrac{1}{2}\nabla_a R\nabla^a R - \tfrac{1}{2}R^2(\nabla_a\psi + eA_a)(\nabla^a\psi + eA^a) - \frac{\lambda}{8}(R^2-\eta^2)^2 - \tfrac{1}{4}F_{ab}F^{ab} \tag{1}$$

where $F_{ab} = \nabla_a A_b - \nabla_b A_a$; e, λ, and η are constants; and $m_A^2 = e^2\eta^2$ and $m_\Phi^2 = \lambda\eta^2$ are the masses of the vector and scalar fields, respectively. In the above expression e and λ are the coupling constants concerning the $U(1)$ and scalar fields, and η defines the energy scale of a symmetry breaking. For most grand unification scales $\eta \simeq 10^{-4}$. In these cases the thickness of the string is $\delta \simeq 10^{-28}$ cm and $\mu \simeq 10^{-7}$. On the electroweak scale $\delta \simeq 10^{-15}$ cm and $\mu \simeq 10^{-33}$. The term $V(|\Phi|) = \lambda(\Phi^2 - \eta^2)^2$ in equation (1) is an effective potential which is supposed to have axial symmetry, i.e., $V = V(R)$.

To deal with Einstein's equations, a static and cylindrically symmetric space-time metric is assumed

$$ds^2 = -e^A\, dt^2 + e^B\, dz^2 + e^C\, d\phi^2 + d\rho^2 \tag{2}$$

where A, B, and C are functions of the radial coordinate ρ.

For a vacuum string there is a barrier in the effective potential $V(|\Phi|)$ between the false vacuum $|\Phi| = 0$ and the global vacuum $|\Phi| = \eta$. The ratio $\alpha = m_A/m_\Phi = e/\sqrt{\lambda}$ measures the comparison between the radii of the core false vacuum and the magnetic field tube.

Now we assume that the values of the scalar and $U(1)$ gauge fields have no significant variations. At the core of the string, any change in the values of the field variables takes place at a cylindrical shell whose thickness is much smaller than the radius of the string.

With the normalization of the radial coordinate given by

$$r=\sqrt{\lambda}\,\eta\rho=m_\Phi\rho=m_A\alpha^{-1}\rho \tag{3}$$

the radius of the core false vacuum is $r_\Phi\simeq 1$ and the radius of the magnetic field tube is $r_A\simeq\alpha^{-1}$.

Since a cosmic string is a configuration of scalar and gauge fields, to find its gravitational effects it is not sufficient to consider the stress energy and solve for the metric. In order to be consistent, we have to find the metric by solving simultaneously the coupled Einstein-scalar-gauge equations. With the normalization given in equation (3) and the definitions

$$A_a=\frac{1}{e}P(\rho)\nabla_a\psi-\frac{1}{e}\nabla_a\psi \tag{4a}$$

$$X=\eta^{-1}R \tag{4b}$$

$$K=m_\Phi\,e^{A+C/2}=\sqrt{\lambda}\,\eta H \tag{4c}$$

the field equations are the coupled nonlinear differential equations (Garfinkle and Laguna, 1989)

$$(KA')'-4\pi\eta^2\left[-\frac{K}{2}(X^2-1)^2+2\alpha^{-2}K^{-1}\,e^{2A}(P')^2\right]=0 \tag{5}$$

$$K''-4\pi\eta^2[-2K^{-1}\,e^{2A}P^2X^2-\tfrac{3}{4}K(X^2-1)^2+\alpha^{-2}K^{-1}\,e^{2A}(P')^2]=0 \tag{6}$$

$$K(KX')'-X[\tfrac{1}{2}K^2(X^2-1)+e^{2A}P^2]=0 \tag{7}$$

$$K(e^{2A}K^{-1}P')'-\alpha^2\,e^{2A}X^2P=0 \tag{8}$$

where the prime denotes d/dr.

3. NUMERICAL RESULTS AND CONCLUSIONS

To search numerical solutions for equations (5)–(8) in the case of grand unification scales, we have imposed the following boundary conditions:

(i) At the core of the string ($r=0$), the scalar and vector fields must be null, which leads to $X(0)=0$, $P(0)=1$, and $V(\Phi)=\lambda\eta^4$.
(ii) Normalization of a Killing basis $[(\partial_t)^a, (\partial_z)^a, (\partial_\phi)^a]$ provides the boundary conditions $A(0)=B(0)=0$ for the metric fields along the z axis and also $\lim_{\rho\to 0}=\rho^2$, leading to the Minkowski metric.

Furthermore, the fields A and B must be equal everywhere, i.e., the string possesses explicit Lorentz invariance along its axis.

(iii) With the above conditions and the definitions given in equations (4), it is easy to see that $K(0)=0$.

(iv) Since $d\Phi/dr=e^{i\phi}\eta X'$ must be null as $r\to\infty$, we impose $X'(\infty)=0$.

(v) The effective potential and the magnetic field must be null at $r=\infty$; then $X(\infty)=1$ and $P(\infty)=0$.

(vi) A Coulomb gauge-like condition $\nabla\cdot A_a=0$, demanded at $r=0$ in equation (4a), leads to $P'(0)=0$.

(vii) At the surface of the string, $V(\Phi)=\beta\lambda\eta^4$, where $\Phi=\Phi_0$ $(0<\Phi_0<\eta)$ and $\beta\geq 1$.

(viii) To guarantee that near the z axis the space-time metric is smooth and that the normalization of the Killing field $(\partial_\phi)^a$ is preserved, we must have $e^C(0)=0$ and $\lim_{\rho\to 0}(d/d\rho)e^C(0)=1$. This leads, after a straightforward calculation, to $K'(0)=1$.

(ix) To recover the Minkowski metric far from the string axis, we must have $A(\infty)=0$.

(x) For a weak-field approximation $\eta^2\to 0$, equations (5) and (6) give $K=r$, $A=0$, and $A'(0)=0$. These values lead from equations (7) and (8) to Nielsen and Olesen's (1978) equations.

Although we have argued that solutions exist, nothing can be said about their stability. The numerical technique we employed was the finite-difference method. First, an expansion of the fields X and P around the point $r=0$ was supposed, by means of

$$
\begin{aligned}
X&=a_1r+a_2r^2+a_3r^3+a_4r^4+a_5r^5+a_6r^6\\
P&=1-b_2r^2+b_3r^3+b_4r^4+b_5r^5+b_6r^6+b_7r^7
\end{aligned}
\tag{9}
$$

leading to $a_1=X'(0)$ and $b_2=-\frac{1}{2}P''(0)$. Imposing the conditions $X(0)=0$, $P(0)=1$, and $P'(0)=0$, we searched for solutions such that for $r\to\infty$

$$X=1-C\,e^{-r}$$

and (10)

$$P=c\,e^{-\sqrt{c}\,r}$$

by expanding up to $r=9$ [notice that Garfinkle (1985) considered the range 0–3, assuming $\alpha=1.0$ in a different technique]. All other boundary conditions mentioned before for X and P have been taken into account here. The same technique as above was used to determine A and K. The polynomial forms are given below for different values of α:

(i) $\alpha = 0.5$:

$$\begin{aligned} X &= 2.05 \times 10^{-6} r^5 - 6.36 \times 10^{-4} r^4 + 0.0157 r^3 \\ &\quad - 0.15 r^2 + 0.637 r - 0.0275 \\ P &= 1.10 \times 10^{-4} r^5 - 3.05 \times 10^{-3} r^4 - 0.031 r^3 \\ &\quad - 0.124 r^2 + 4.62 \times 10^{-3} r + 1.0 \\ A &= (-1.98 \times 10^{-4} r^6 + 6.04 \times 10^{-3} r^5 - 0.0708 r^4 \\ &\quad + 0.387 r^3 - 0.878 r^2 - 4.21 \times 10^{-3} r + 0.027) \times 10^{-8} \\ K &= r \end{aligned} \tag{11}$$

(ii) $\alpha = 1.0$:

$$\begin{aligned} X &= -8.84 \times 10^{-4} r^4 + 0.021 r^3 - 0.188 r^2 + 0.716 r - 0.013 \\ P &= 2.570 \times 10^{-4} r^5 - 6.180 \times 10^{-3} r^4 + 0.050 r^3 \\ &\quad - 0.132 r^2 - 0.196 r + 1.05 \\ A &= (3.510 \times 10^{-5} r^6 - 1.590 \times 10^{-3} r^5 + 0.027 r^4 - 0.224 r^3 \\ &\quad + 0.867 r^2 - 0.703 r + 0.129) \times 10^{-15} \\ K &= r \end{aligned} \tag{12}$$

(iii) $\alpha = 1.5$:

$$\begin{aligned} X &= -1.300 \times 10^{-7} r^6 + 1.390 \times 10^{-4} r^5 - 4.200 \times 10^{-3} r^4 + 0.050 r^3 \\ &\quad - 0.296 r^2 + 0.868 r - 0.029 \\ P &= 2.620 \times 10^{-5} r^7 - 9.350 \times 10^{-4} r^6 + 0.013 r^5 - 0.099 r^4 \\ &\quad + 0.386 r^3 - 0.672 r^2 + 0.030 r + 1 \\ A &= (-1.010 \times 10^{-4} r^5 - 7.620 \times 10^{-3} r^4 + 0.217 r^3 \\ &\quad - 1.940 r^2 + 7.130 r - 1.210) \times 10^{-9} \\ K &= r \end{aligned} \tag{13}$$

A comparison between the main analytical and numerical values for the above solutions is shown in Table I. Numerical values corresponding to $r \to \infty$ have been considered for $r = 9$. The plots of these solutions are shown in Figures 1–5.

Table I

Analytical value	Numerical value	$\alpha=0.5$	$\alpha=1$	$\alpha=1.5$
$X(0)=0$	$X(0)$	-0.0275	-0.0133	-0.0292
$X(\infty)=1$	$X(9)$	0.9490	0.9304	0.9122
$X'(\infty)=0$	$X'(9)$	-0.0352	-0.0698	-0.0190
$P(0)=1$	$P(0)$	1	1.05	1
$P'(0)=0$	$P'(0)$	0.0046	-0.1960	0.0305
$P(\infty)=0$	$P(9)$	0.0809	-0.0357	0.3388
$A(0)=0$	$A(0)$	2.77×10^{-10}	1.29×10^{-16}	-1.21×10^{-9}
$A'(0)=0$	$A'(0)$	-4.21×10^{-11}	-7.03×10^{-16}	7.13×10^{-9}
$A(\infty)=0$	$A(9)$	-2.09×10^{-8}	3.95×10^{-15}	8.05×10^{-9}
$K(0)=0$	$K(0)$	0	0	0
$K'(0)=1$	$K'(0)$	1	1	1
$V(0)=\lambda\eta^4$	$V(0)$	$0.9984\lambda\eta^4$	$0.9996\lambda\eta^4$	$0.9982\lambda\eta^4$
$V(\infty)=0$	$V(9)$	$0.0098\lambda\eta^4$	$0.0180\lambda\eta^4$	$0.0281\lambda\eta^4$

The deficit angle produced by the string is given in terms of its linear density μ by (Garfinkle, 1985)

$$\begin{aligned}\Delta\phi&=8\pi\mu+\frac{\pi}{2}\int_0^\infty e^{-A}H\left(\frac{dA}{d\rho}\right)^2 d\rho\\&=8\pi\mu+\frac{\pi}{2}\int_0^\infty e^{-A}K(A')^2\,dr=8\pi\mu+\pi\delta_2\end{aligned}\tag{14}$$

where δ_2 is its second-order term. To determine this term numerically, we have expanded e^{-A} up to the fourth order and performed the above integral from 0 to 9, considering the different values of α.

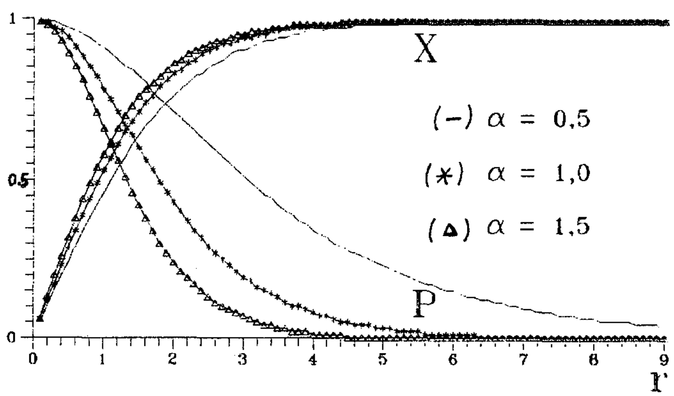

Fig. 1. Magnitudes of the scalar and gauge fields as functions of r for different values of α.

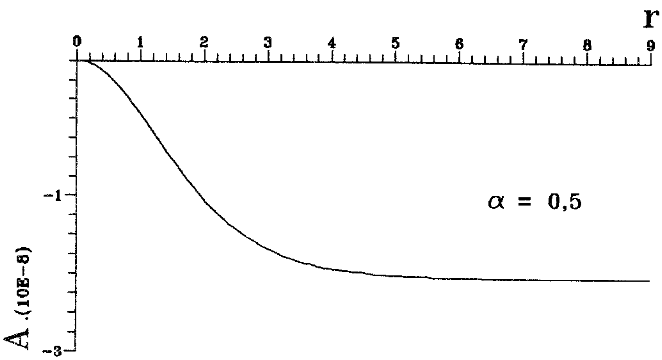

Fig. 2. Plot of A as a function of r for $\alpha = 0.5$.

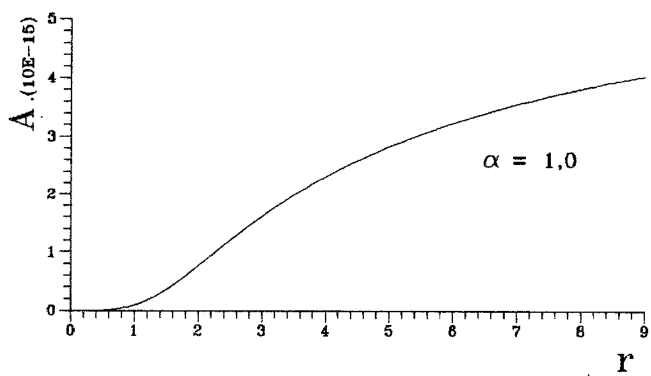

Fig. 3. Plot of A as a function of r for $\alpha = 1.0$, showing the small numerical change of A in this case.

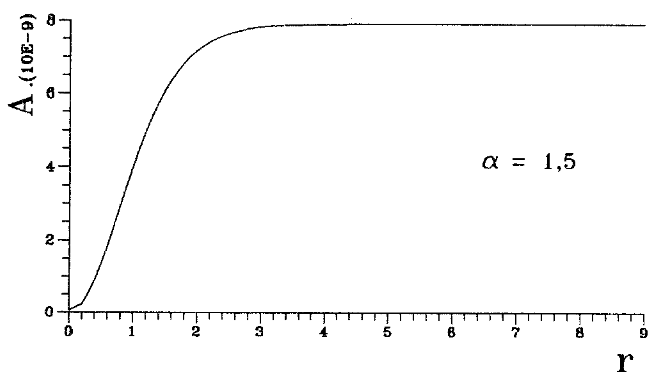

Fig. 4. Plot of A as a function of r for $\alpha = 1.5$.

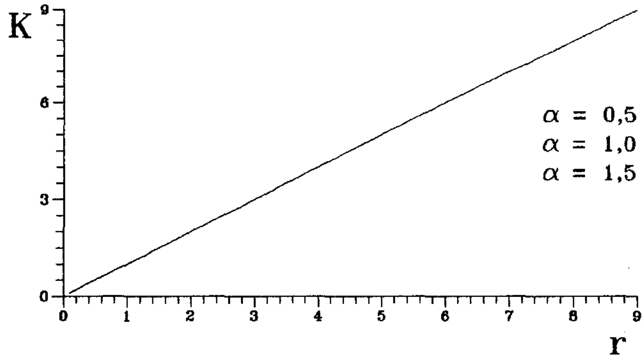

Fig. 5. Plot of K as a function of r for different values of α.

This leads to the results

(i) $\alpha=0.5$ $\qquad\delta_2=1.028\times 10^{-16}$

(ii) $\alpha=1.0$ $\qquad\delta_2=3.856\times 10^{-30}$

(iii) $\alpha=1.5$ $\qquad\delta_2=1.339\times 10^{-17}$

The above corrections for $\alpha=0.5$ and $\alpha=1.5$ are of the same order of η^4, as pointed out by Garfinkle and Laguna (1989). However, for the transition case ($\alpha=1.0$), A must be a constant, according to Laguna-Castilho and Matzner (1987*b*), which leads to $\delta_2=0$. Our numerical results show clearly for A a change of about 10^{-15} in the range 0–9 leading to a second-order correction $\delta_2\sim 10^{-30}$ for $\Delta\phi$, which is reasonable. Moreover, the functions A and K are near their flat-space values for small η in the above case.

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